

A VARIATIONAL PRINCIPLE IN KREIN SPACE

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ABSTRACT. A variational characterization, involving a max-inf of the Rayleigh quotient, is established for certain eigenvalues of a wide class of definitizable selfadjoint operators Q in a Krein space. The operator Q may have continuous spectrum and nonreal and nonsemisimple eigenvalues; in particular it may have embedded eigenvalues. Various applications are given to selfadjoint linear and quadratic eigenvalue problems with weak definiteness assumptions.

1. INTRODUCTION

Variational characterizations of eigenvalues have been studied for over a century and have been applied to comparison principles and numerical algorithms for a wide range of problems. For example, the generalised eigenvalue problem

$$(1.1) \quad Ax = \lambda Bx$$

with selfadjoint operators A and B in a Hilbert space H , can be studied in the “right definite case” (where B is positive definite) via the equation

$$(1.2) \quad B^{-1}Ax = \lambda x.$$

Formally, $B^{-1}A$ is selfadjoint in a “weighted” Hilbert space H_B with inner product generated by

$$(1.3) \quad (x, y)_B = (x, By)$$

and the variational principles that we consider involve the Rayleigh quotient

$$(1.4) \quad r(x) = \frac{(x, B^{-1}Ax)_B}{(x, x)_B} = \frac{(x, Ax)}{(x, Bx)}$$

which can be evaluated directly from the operators A and B . However, if B is not definite then (1.4) vanishes for some nonzero x , so the usual variational principles need modification. If A is definite (this is the so-called “left definite” problem) then (1.2) can be replaced by $A^{-1}Bx = \lambda^{-1}x$ in a Hilbert space H_A .

Received by the editors March 25, 1991.

1991 *Mathematics Subject Classification.* Primary 49G05, 47B15, 35J20.

Key words and phrases. Minimax principle for eigenvalues, selfadjoint operators in Krein space, spectral functions, elliptic problems with indefinite weight.

Research of the first author supported by NSERC of Canada.

Research of the second author supported in part by the Ministry of Science of Croatia and was carried out while he was visiting the Department of Mathematics and Statistics, University of Calgary.

See, e.g., [20] for a variational treatment of various boundary value problems by these techniques.

Problems which are neither left nor right definite have attracted considerable attention in recent years (cf. [9, 12] and their references), and it turns out that they have properties different from those of definite cases. The variational characterization of eigenvalues in such problems has however remained open despite early work of Richardson [16] for left definite, and a conjecture [17] for general, Sturm-Liouville problems, that the solution “for m large enough, may be regarded as furnishing a minimum of a calculus of variations problem”. Richardson does not explicitly state which problem, but it is clear from the context that he means minimization of (x, Ax) subject to $(x, Bx) = \pm 1$ and m (recursively defined) linear constraints of the form $(y_i^\pm, Bx) = 0$. Nor is he clear about “large enough”, but he cites his [17, §4] which shows that, for large enough m , there are precisely two eigenfunctions y_m^\pm which oscillate m times (one for each sign of λ —see [17, Theorem VII et prec.]). Our results show that each such eigenfunction obeys a double extremum principle, including the above recursive one.

If B is 1-1 then H_B , with the inner product generated by (1.3), becomes a Krein space. An early variational principle in this setting is due to Phillips [14]. It characterizes eigenvalues of a positive compact operator in H_B by triple extrema involving sign-definite subspaces and thus it implicitly involves the positive and negative cones

$$(1.5) \quad C^\pm = \{x \in H_B : \pm(x, x)_B > 0\}.$$

This theory was extended to noncompact nonnegative operators by Textorius [18]. Various authors have also used these cones explicitly; for example, it was noted in [1] that (single) extrema of r in (1.4) over C^\pm give principal eigenvalues for certain left definite problems. Double extrema involving C^\pm have been used in [10 and 13] for some of the real eigenvalues λ for indefinite matrix problems (1.1). Slightly different double (and also triple) extremal characterizations were established in [2] for elliptic equations with indefinite weights. Here we shall use spectral functions of definitizable selfadjoint operators in Krein spaces [11] to establish a double extremum principle for some of the eigenvalues under conditions which allow continuous spectrum, and in particular embedded eigenvalues corresponding to critical points (see §2). Our work includes all previous results that we know of.

In §2 we introduce our assumptions and state the main result, which is proved in §3. We conclude with §4 where applications to differential operators are discussed. In particular, we give a detailed comparison with [2] which uses two parameter ideas for statements and proofs, and we also show how some asymptotic estimates of [8] may be extended from the left definite case.

We conclude this section with some brief remarks on the application of Theorem 2.3 to problems of the form (1.1). Our assumptions are satisfied if, for example, A is bounded below with compact resolvent and B is injective and A -form compact, cf. [15]. Indeed, replacing A by $A - \nu B$ for some real ν if necessary, we see that the space H_A is a Pontryagin space and the verification of our hypotheses is straightforward.

The selfadjoint quadratic eigenvalue problem

$$(1.6) \quad (\mu^2 R + \mu S + T)y = 0, \quad 0 \neq y \in X,$$

can be transformed to (1.1) (see [9]) by defining $H = X \times X$, $\lambda = 1/\mu - \mu$,

$$A = \begin{bmatrix} S & -R - T \\ -R - T & S \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -T & 0 \\ 0 & R \end{bmatrix}$$

on appropriate domains. If S is bounded below with compact resolvent and T, R are injective S -form compact operators then the above remarks lead to a max-inf characterization of eigenvalues of the problem (1.6). The above setting includes the well-known overdamped case from continuum mechanics which leads to a left-definite problem. We also cite Weinberger's Rayleigh quotient variational principle [19] for a quadratic problem with right-definite linearization.

2. BACKGROUND AND BASIC RESULTS

2.1 Cones. Throughout §§2 and 3, Q is a selfadjoint operator in a Krein space K , with a (generally indefinite) inner product $[\cdot, \cdot]$ —cf. [3 and 7] for basic terminology etc.

Let C be a cone in K , so $\lambda C \subseteq C$ for all $\lambda > 0$. We define the *internal dimension* $d(C)$ to be the supremum of the dimensions of all subspaces in $C \cup \{0\}$. As in (1.5) we define the *positive* and *negative cones* of K by

$$(2.1) \quad C^\pm = \{x \in K : \pm[x, x] > 0\}, \quad \text{with } C_0^\pm = C^\pm \cup \{0\}.$$

Note that K is a Pontryagin space if at least one of $d(C^\pm)$ is finite. For any real number ν we also define the cone

$$\Gamma_\nu^- = \{x \in D(Q) : [x, Q_\nu x] \leq 0\}$$

where $Q_\nu := Q - \nu I$.

We shall make three assumptions on Q , the first being

- (I) For some real ν , (α) Q_ν is (boundedly) invertible
and (β) $d(\Gamma_\nu^-)$ is finite.

Assumption $(I\alpha)$ guarantees that $\sigma(Q)$ does not contain \mathbb{R} . If $\sigma(Q)$ contains \mathbb{R} , then the variational principle below is vacuous. Also, $(I\beta)$ means that $[x, Qx] - \nu[x, x]$ has finitely many negative squares on $D(Q)$.

When $\sigma(Q)$ is discrete, our preparations are relatively simple and we shall discuss this case first. In particular, our remaining assumptions (and $(I\alpha)$) are automatic.

Let $G_\lambda = N(Q_\lambda)$ be the *geometric eigenspace* at λ . If $G_\lambda \subset C_0^+$ (resp. C_0^-), we say that λ is of *positive* (resp. *negative*) *type*. Here C_0^\pm come from (2.1). Developing this, we define the *positive and negative multiplicities* of an eigenvalue λ by $d^\pm(\lambda) = d(G_\lambda \cap C^\pm)$. Writing N^+ (resp. N^-) for the set of eigenvalues which are *not* of positive (resp. negative) type, we define

$$(2.2) \quad \theta_p = \sup(N^+ \cap \mathbb{R}).$$

Recall that, for any eigenvalue λ , the *algebraic eigenspace* A_λ is the span of the *Jordan chains* (y_1, \dots, y_l) of *length* l , where $0 \neq y_1 \in G_\lambda$ and

$$(2.3) \quad Q_\lambda y_j = y_{j-1}, \quad 2 \leq j \leq l.$$

It is easily seen that $[y_1, y_1] = 0$ if $l > 1$ or if $\lambda \notin \mathbb{R}$, so such $\lambda \in N^+ \cap N^-$. Moreover it follows from $(I\beta)$ that the sum of the lengths l , over such λ , must be finite.

Defining the subspace F as the sum of the A_λ for all $\lambda \in N^-$, and writing θ for θ_p , which is now $\max(N^+ \cap \mathbb{R})$, we may then proceed directly to Proposition 2.2 below.

2.2 Spectral functions with critical points. We proceed now to discuss the case when $\sigma(Q)$ is not discrete. For the reader's convenience, we have assembled some basic definitions and properties related to definitizable selfadjoint operators on Krein spaces. For further details see [3, §4.1], [7, §VIII.6] and especially [11]. A selfadjoint operator Q is definitizable if its spectrum $\sigma(Q)$ is not the whole complex plane, and if a nonconstant *definitizing polynomial* π exists so that $\pi(Q)$ is nonnegative definite. Such an operator has a *spectral function* E , i.e., an orthoprojector-valued function defined on real intervals whose endpoints are not zeros of π , together with the complements of such intervals in \mathbb{R} . Then $\pm E(\Delta)$ is positive definite if $\pm \pi$ is positive on $\Delta \cap \sigma(Q)$, and

$$Qx = \int_{\Delta} \lambda dE(\lambda)x \quad \text{for all } x \in E(\Delta)K.$$

Here and below, Δ will always denote a set on which E is defined as above. E may be extended to further subsets of \mathbb{R} , but the above definition suffices for our purposes. We note that

$$(2.4) \quad \sigma(Q|_{E(\Delta)K}) \subset \overline{\Delta}$$

so $Q|_{E(\Delta)K}$ is invertible if $0 \notin \overline{\Delta}$.

Proposition 2.1. Q is definitizable, and hence admits a spectral function E .

Proof. In the case $\nu = 0$, this is a result of Langer [11], who constructs an “equivalent” selfadjoint operator on an auxiliary Pontryagin space. In our case we use Q_ν instead of Q , and we obtain a definitizing polynomial of the form

$$(2.5) \quad \pi(\lambda) = (\lambda - \nu)|p(\lambda)|^2$$

for some polynomial p . \square

We term $\lambda \in \mathbb{R} \cup \{\infty\}$ a *critical point* of Q , and we write $\lambda \in c(Q)$, if $E(\Delta)$ is indefinite whenever $\lambda \in \text{int} \Delta$. Necessarily then $\lambda \in \sigma(Q)$. Also $\infty \neq \lambda \in c(Q)$ implies $\pi(\lambda) = 0$ by the above, so there are only finitely many critical points. A critical point is *regular* if there are $\alpha > 0$ and a neighbourhood Δ_0 of λ so that $\|E(\Delta)\| \leq \alpha$ whenever $\lambda \in \Delta \subset \Delta_0$. This is the case, for example, if λ is an isolated point of $\sigma(Q)$.

We define $\lambda \in \sigma(Q)$ to be of *positive* (resp. *negative*) type if $E(\Delta)K \subset C_0^+$ (resp. C_0^-) for some Δ containing λ . This extends our previous definition to nonisolated λ . As before, $\lambda \in N^+$ (resp. N^-) if λ is not of positive (resp. negative) type.

Note that ν is not a critical point of Q by (I α), but that ∞ is a critical point in general. Using (2.5) we see that $\lambda > \nu$ (resp. $< \nu$) is a finite critical point iff $p(\lambda) = 0$ and λ is an eigenvalue not of definite type, i.e., $\lambda \in N^+ \cap N^-$.

Our next assumption is

(II) *The finite critical points of Q are all regular.*

Regularity is automatic for isolated eigenvalues. Suppose now that all eigenvalues of positive type exceed ν . In this case we define θ as the greater of θ_c and θ_p , where θ_c is the supremum of the continuous spectrum $< \nu$, and θ_p

comes from (2.2). It follows from (II) that $\sum A_\lambda$ is nondegenerate (the sum being over $\lambda \in N^-$) but this is in general infinite dimensional. The following refinement uses an orthogonal decomposition $A_\lambda = F_\lambda[+]H_\lambda$ given in [6], where F_λ is finite dimensional and nondegenerate, and where $H_\lambda \subseteq G_\lambda$ is a Hilbert space under $[\cdot, \cdot]$. Specifically, defining F as the sum of the A_λ for nonreal λ and F_λ for real $\lambda \in N^-$, we may proceed to Proposition 2.2.

Finally, suppose $\theta > \nu$, so θ is simply the maximum (by (I β)) of the eigenvalues $\lambda > \nu$ with $\lambda \in N^+$. In this case we need a third assumption

(III) $\sigma(Q) \cap [\nu, \theta]$ consists of finitely many eigenvalues of finite multiplicity.

We remark, by virtue of (I β) and our previous comments, that multiplicity here could be interpreted as “positive”, “geometric” or “algebraic”. To the previous definition of F we then add the sum of the A_λ over $\lambda \in [\nu, \theta]$.

Proposition 2.2. *F defined as above, is finite dimensional, nondegenerate and invariant for Q . Moreover all real eigenvalues of $Q|_F$ are $\leq \theta$, while $Q|_{F^\perp}$ has real spectrum, of negative type $< \theta$ and of positive type $> \theta$.*

Proof. The only nontrivial contention is nondegeneracy of F , which is covered by the arguments of [6, §5]. \square

We remark that the results of Propositions 2.1 and 2.2 could be used instead of (I)–(III).

2.3 A variational principle. We list those eigenvalues $\lambda > \theta$, for which $d^+(\lambda) > 0$, in nondecreasing order and counted by positive multiplicity, as $\lambda_1^+ \leq \lambda_2^+ \leq \dots$. Our main result states that these eigenvalues may be characterized by a max-inf principle with shifted index. More precisely, recalling Proposition 2.2 and defining $d^\pm = d(F \cap C^\pm)$ and

$$(2.6) \quad \mu_j = \max\{\inf\{[x, Qx]/[x, x] : x \in M \cap C^+ \cap D(Q)\} : M \in \mathcal{M}_j\}$$

where \mathcal{M}_j is the set of subspaces of K of codimension $j - 1$, we have

Theorem 2.3. *Assume (I)–(III). Then $\lambda_j^+ = \mu_{j+d^+}$, $j = 1, 2, \dots$.*

Remark 2.4. If C^+ is replaced by C^- in (2.6) then corresponding eigenvalues λ_j^- with eigenvectors in C^- can be characterized. Similarly, if the definition of θ and (II) are modified in an obvious way then λ_j^\pm may be listed in nonincreasing order and all those below a certain θ' can be characterized.

Remark 2.5. When $\dim K$ is finite, Theorem 2.2 is equivalent to a result in [13]. In fact the setting of (1.1), where $K = H_B$ and $Q = A^{-1}B$ with B invertible, is equivalent to the one here. Note that $[x, Qx]/[x, x] = r(x)$ of (1.4) while (I)–(III) are automatic in finite dimensions. Also the other formulae for λ_j^\pm mentioned in Remark 2.4 can be read from [13] (with dimensions there replaced by appropriate codimensions).

Similarly, in the left-definite case of (1.1), Q is a uniformly positive operator in K and so (I)–(III) are automatic with $\theta = \nu = 0$. In the right-definite case we recover the ordinary Weyl-Courant principle in a Hilbert space.

Remark 2.6. For any Jordan chain (y_1, \dots, y_l) (see (2.3)), an easy calculation shows that $\delta = [y_1, y_l]$ is real. If $\delta < 0$ then we say that the chain has *negative sign* ($\operatorname{sgn} \delta$ is associated with various terms in the literature, e.g. inertial signature and sign characteristic). It turns out that negative sign Jordan chains

of length $l = 2$ may be ignored in the calculation of θ . This improves Theorem 2.3 and follows from arguments in [13] and §3.

Remark 2.7. If the λ_j^+ cease to exist from some j on, the remaining μ_{j+d^+} characterize the infimum of the continuous spectrum $> \theta$.

3. PROOF OF THE VARIATIONAL PRINCIPLE

For notational ease we fix j and write $\lambda = \lambda_j^+$.

Lemma 3.1. *There are d^\pm -dimensional subspaces F^\pm of F such that $F^+ \subset C_0^+$ and*

$$(3.1\pm) \quad \pm[x, Q_\lambda x] \leq 0 \quad \text{for all } x \in F^\pm.$$

Proof. Since $\lambda > \theta$, Proposition 2.2 shows that $Q|_F$ has no (real) eigenvalues $> \lambda$. The result now follows from [13, Steps 1 and 2], which applies since F is nondegenerate. \square

We shall base the proof on the following elementary decomposition principle.

Lemma 3.2. *If (i) $[y, z] = 0 = [y, Qz]$ for all $y \in Y$, $z \in Z$ where $Y, Z \subseteq D(Q)$ and (ii) (3.1 \pm) holds for all $x \in Y$ and all $x \in Z$, then (3.1 \pm) holds for all $x \in Y + Z$.*

Now let $x_l \in C^+$ satisfy $[x_l, x_l] = 1$ and form a linearly independent set of eigenvectors for λ_l^+ , $l = 1, \dots, j$, and write S_j for the span $\text{sp}\{x_1, \dots, x_j\}$.

Lemma 3.3. $\theta_{d^++j} \leq \lambda$.

Proof. Since $S := F^+ + S_j$ is a $(d^+ + j)$ -dimensional subspace of C_0^+ , for any $M \in \mathcal{M}_{d^++j}$ there is $x \in M \cap S \cap C^+$. Evidently (3.1+) holds on S_j , so by Lemmas 3.1 and 3.2, (3.1+) holds on S and $\mu_{d^++j} \leq [x, Qx]/[x, x] \leq \lambda$. \square

Since $S_{j-1} = \text{sp}\{x_1, \dots, x_{j-1}\} \subset C_0^+$, $F + S_{j-1}$ is nondegenerate, and hence has an orthocomplement T in K [7, Corollary I.11.9]. Let E_T be the spectral function of $Q|_T$. Fix $\omega > 0$ so that $\Omega :=] - \omega, \omega[$ contains λ, ν and the zeros of p (2.5).

Lemma 3.4. (3.1-) holds for all $x \in E_T(\Omega)K$.

Proof. Since $\lambda > \theta$, Proposition 2.2 shows that $[x, E_T(\cdot)x]$ is nondecreasing on $] \lambda, \infty[$. Similarly the construction of F and the definition of T show that $[x, E_T(\cdot)x]$ is nonincreasing on $] - \infty, \lambda[$. Thus

$$\int_{-\omega}^{\omega} \mu[x, E(d\mu)x] \geq \lambda \int_{-\omega}^{\omega} [x, E(d\mu)x],$$

whence $[x, Qx] \geq \lambda[x, x]$, for any $x \in E_T(\Omega)K$. \square

Lemma 3.5. (3.1-) holds for all $x \in PK$ where $P = E(\overline{\mathbb{R}} \setminus \Omega)$.

Proof. For all real $t \notin \Omega$, $\varphi(t) := (t - \lambda)/(t - \nu) > 0$. By (2.5), $[x, Q_\nu x] \geq 0$ for all $x \in R(p(Q))$ and so

$$(3.2) \quad [Q_\lambda x, x] = [\varphi(Q)Q_\nu x, x] \geq 0$$

whenever $x = p(Q)y$ with $y \in D(p(Q)P)$. Noting that $p(\lambda)$ is bounded away from zero for all $\lambda \notin \Omega$, we see that $p(Q)P|_{PK}$ is invertible, cf. (2.4), so (3.2) holds for all $x \in PK$. \square

We are now in a position to complete the proof of Theorem 2.3. By Lemmas 3.2, 3.4, and 3.5, (3.1-) holds on $E_T(\mathbb{R})K = T$. By Lemmas 3.1 and 3.2, then, (3.1-) holds on

$$(3.3) \quad M_j = F^- + T$$

which belongs to \mathcal{M}_{d^++j} since F is nondegenerate. Thus

$$\mu_{d^++j} \geq \inf\{[x, Qx]/[x, x] : x \in M_j \cap C^+\} \geq \lambda,$$

which, with Lemma 3.3, establishes our theorem. \square

Remark 3.6. If N is a subspace of K such that $M_j \cap N = \{0\}$, where M_j is the maximising subspace of (3.3), then Theorem 2.3 gives

$$\lambda_j^+ = \max\{\inf\{[x, Qx]/[x, x] : x \in M \cap D(Q) \cap C^+\} : M \in \mathcal{M}_{j+d^+}, M \cap N = \{0\}\}.$$

Since [13] one can ensure that $F^+ \cap F^- = \{0\}$, we can choose

(i) $\dim N = d^+ - \sum_{\lambda \leq \theta} d^+(\lambda)$, and

(ii) $N \subseteq F$,

so N is spanned by root vectors of $Q|_F$.

This will be used in 4.3. Note that (i) cannot be relaxed to $\dim N > d^+$.

4. ELLIPTIC EQUATIONS WITH INDEFINITE WEIGHTS

4.1 Setting. Let Ω be a bounded smooth domain in \mathbb{R}^n and $q, w \in L^\infty(\Omega)$. We consider the eigenvalue problem

$$(4.1) \quad (-\Delta + q)y = \lambda wy, \quad y = 0 \quad \text{on } \partial\Omega.$$

As in [2], a (nonzero) function $y \in H := W_0^{1,2}(\Omega)$ satisfying (4.1) is an eigenfunction for the eigenvalue $\lambda \in \mathbb{C}$; moreover, $-\Delta + q$ can be replaced by a uniformly elliptic operator and the Dirichlet condition by a more general self-adjoint boundary condition. Actually our Krein space setting is sufficiently general to handle continuous spectrum (thus allowing unbounded Ω and/or singular coefficients and weight function w). We adhere to the problem (4.1) and the above restrictive assumptions on Q and w for simplicity and also to facilitate a detailed comparison with [2]. We impose the assumption

$$(4.2) \quad \{x : w(x) = 0\} \text{ is of Lebesgue measure zero}$$

which is more stringent than the corresponding assumption of [2]; on the other hand we do not need the unique continuation assumption of [2]. The only reason we impose (4.2) is to form the Krein space K below, and moreover we hope to show elsewhere how to remove (4.2) entirely.

Let L be the selfadjoint realization of $-\Delta + q$, with Dirichlet boundary condition, in $L^2(\Omega)$, so $D(L) = W^{2,2}(\Omega) \cap H$. From [5, Lemma 2.1] we can translate the λ -origin if necessary to ensure that L has a compact inverse. Note also that L is semibounded, and the bilinear form associated with L is given by

$$l(y, z) = \int_{\Omega} (\nabla y \cdot \nabla \bar{z} + qy\bar{z}) \quad \text{on } D(l) = H.$$

4.2 Application of Theorem 2.3. Let K be the space $L^2_{|w|}$ (consisting of functions y for which $\|y\|_{|w|}^2 = \int_{\Omega} |w| |y|^2$ is finite) endowed with the (possibly indefinite) inner product

$$[y, z] = \int_{\Omega} w y \bar{z}.$$

Then K is a Krein space. Since $w \in L^{\infty}(\Omega)$, it is clear that $L^2(\Omega)$ is continuously embedded in K . Let

$$D(Q) = \{y \in D(L) : |w|^{-1/2} Ly \in L^2(\Omega)\}, \quad Qy = w^{-1} Ly \quad \text{if } y \in D(Q).$$

Clearly Q is an operator in K . In fact we have

Theorem 4.1. *The operator Q is a semibounded selfadjoint operator in K with compact inverse.*

Proof. First if $x_j \rightarrow 0$ in K then $Q^{-1}x_j = L^{-1}wx_j \rightarrow 0$ in $L^2(\Omega)$, hence in K . Thus Q is boundedly invertible. Next we show that

(4.3) Q is densely defined in K .

Assume x is orthogonal to $D(Q)$ in K . Then

$$[x, Q^{-1}y] = \int_{\Omega} w L^{-1}(wy) \bar{x} = 0 \quad \text{for all } y \in K.$$

Since $(\operatorname{sgn} w)|w|^{1/2}$ is an isomorphism of K onto $L^2(\Omega)$, the selfadjointness of L shows that $|w|^{1/2}L^{-1}(wx) = 0$ a.e. in Ω , whence $x = 0$.

For any $y, z \in D(Q)$ we have $[Qy, z] = \int_{\Omega} (Ly) \bar{z}$ so Q is symmetric and semibounded since $D(Q) \subset D(L)$. Let $z \in D(Q^*)$. Then the functional $\varphi: y \mapsto [Qy, z] = \int_{\Omega} (Ly) \bar{z}$ defined on $D(Q)$ is continuous in K and therefore in $L^2(\Omega)$. An argument similar to the one for (4.3) shows that $D(Q)$ is dense in $H = W_0^{1,2}(\Omega)$ and hence also in $L^2(\Omega)$. As L is selfadjoint we conclude that $z \in D(L)$. Therefore $\varphi: y \mapsto \int_{\Omega} y \overline{(Lz)}$ is continuous in $L^2_{|w|}$, so $z \in D(Q)$, and Q is selfadjoint in K .

Finally let x_j converge weakly to zero in K . Then $w x_j$ converge weakly to zero in $L^2(\Omega)$ so

$$\|Q^{-1}x_j\|_{L^2_{|w|}} = \||w|^{1/2}L^{-1}(wx_j)\|_{L^2} \rightarrow 0,$$

which proves that Q^{-1} is compact. \square

A Jordan chain for (4.1) is a sequence $y_j \in H$, $j = 0, 1, \dots$, such that $y_0 = 0$,

$$(L - \lambda w)y_j = w y_{j-1}, \quad j = 1, \dots,$$

with y_1, \dots linearly independent. We note the following easy result:

Lemma 4.2. *The Jordan chains for (4.1) coincide with those for Q .*

Proof. If $y_1 \in D(Q)$ with $Qy_1 = Ly_1$ then $Ly_1 = \lambda w y_1$ since $D(Q) \subset D(L)$. Conversely, if (4.1) holds then $w^{-1}Ly_1 = \lambda y_1 \in L^2(\Omega)$, so $|w|^{-1/2}Ly_1 \in L^2(\Omega)$, hence $y_1 \in D(Q)$ and $Qy_1 = \lambda y_1$. The argument for $j > 1$ is similar. \square

By Theorem 4.1 and our earlier comments, (I)–(III) hold. Lemma 4.2 we may label the eigenvalues $> \theta$ for (4.1) according to positive multiplicity as $\lambda_1^+ \leq \lambda_2^+ \leq \dots$. Since

$$r(y) = \frac{[y, Qy]}{[y, y]} = \frac{\int_{\Omega} (|\nabla y|^2 + q|y|^2)}{\int_{\Omega} w|y|^2}, \quad y \in D(Q),$$

and since $D(Q)$ is dense in $H = D(I)$, Theorem 2.3 yields the following:

Corollary 4.3. *There exists $\theta \in \mathbb{R}$ so that all eigenpairs (λ, y) of (4.1) satisfying $\lambda = \lambda_j^+ > \theta$, $y \in H$, also satisfy $\int_{\Omega} w|y|^2 > 0$, and moreover*

$$(4.4) \quad \lambda_j^+ = \max \left\{ \inf \left\{ r(z) : \int_{\Omega} w|z|^2 > 0, z \in M \cap H \right\} : M \in \mathcal{M}_{j+d^+} \right\}.$$

4.3 Comparison with [2]. Allegretto and Mingarelli consider a two parameter embedding of (4.1), viz.

$$(4.5) \quad (\Delta + \lambda w + \mu)y = 0$$

and they characterize certain eigenvalues labelled $\lambda_{i,k}$, where $\lambda_{i,k} = \lambda_j^+$ under certain conditions. One condition is that $\lambda_{i,k}$ lies on the k th (λ, μ) eigen-curve of (4.4), i.e., that $\Delta + \lambda w$ has k nonnegative eigenvalues, counted by multiplicity.

Analysis of the eigencurves (cf. [4]) shows that $k = j + d^+$ whenever $\lambda_{i,k} > \theta$ so we may replace (4.4) by

$$(4.6) \quad \lambda_{i,k} = \max \left\{ \inf \left\{ r(z) : \int_{\Omega} w|z|^2 > 0, z \in M \cap H \right\} : M \in \mathcal{M}_k \right\}$$

in Corollary 4.3. This not only shows that the index shift in (4.4) is very natural, but also gives an alternative method for its calculation, involving eigenvalues μ of (4.5) for fixed λ , but not requiring knowledge of F .

[2, Corollary 2.10] contains a formula like (4.6) but with $M \in \mathcal{M}_k$ further constrained by $M \cap N = \{0\}$ where N is a subspace of root vectors such that (i) $\dim N = k - i$, so $\dim N = d^+ - (i - j)$, and (ii) N is neutral, so $N \subseteq F$. Thus [2, Corollary 2.10] follows from (4.6) and Remark 3.6, provided N there can be chosen neutral. This is possible if and only if all Jordan chains of Q of odd length > 1 have negative sign (see Remark 2.6). Moreover if there are no such Jordan chains then N may also be used for similar characterizations of λ_j^- ; cf. Remark 2.4.

We observe in conclusion that $\lambda_{i,k} = \lambda_j^+$ also depends on counting i via positive multiplicities $d^+(\lambda)$, and one must interpret the remarks following [2, Theorem 2.8] in this light.

4.4 Asymptotics. Recently there has been significant interest in eigenvalue asymptotics and “counting functions” for left-definite problems. As an example we cite [8, Theorem 3.1], where the limit

$$(4.7) \quad \lambda_j^+ j^{-2/n} \|w_+\|_{L^{n/2}(\Omega)} \rightarrow (2\pi)^2 (V_n)^{-2/n} \quad \text{as } j \rightarrow \infty$$

is established, with V_n as the volume of the unit ball in \mathbb{R}^n and $w_+ = \max\{0, w\}$. The key tools for this and similar estimates are

(a) corresponding formulae for right definite problems and

(b) variational formulae for eigenvalues of left definite problems. Since (a) is independent of left definiteness and (b) is directly generalised by Corollary 4.3, we conclude

Corollary 4.4. (4.7) carries over to (4.1).

More general problems, as mentioned in 4.1, may also be treated following the remarks in [8].

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